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Prolongation structure without prolongation

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Abstract. A method is proposed for finding a prolongation structure in the Wahlquist–Estabrook sense without using the concept of prolongation. The closure of this structure follows unambiguously from the analysis of a holonomy algebra for a connection in a fibre bundle associated with a given non-linear equation.

Wahlquist and Estabrook (1975, hereafter referred to as WE) proposed a procedure (the pseudopotential method) for finding the Lax pair (Lax 1968) for a given non-linear equation to be solved via the inverse scattering method (Gardner *et al* 1967). The basic element of the pseudopotential method is a representation of a non-linear equation as a set of differential two-forms α_a constituting a closed ideal of forms, with subsequent prolongation of it with a system of one-forms ω^k which depend on auxiliary variables y^k (pseudopotentials). From complete integrability of the Pfaffian system $\omega^k=0$ one obtains some (in general, open) algebraic structure (the prolongation structure). Embedding of this structure into a finite-dimensional Lie algebra or extracting from it a finite-dimensional Lie subalgebra leads to the appearance of a parameter λ which serves as a spectral parameter in the associated linear problem. However, an appropriate effective closure mechanism for the prolongation structure has not been revealed up to now.

In the present paper a method is described for finding the WE-type algebraic structure which does not use the concept of prolongation. An unambiguous algorithmic way for introducing a spectral parameter based on the consideration of a holonomy algebra for a connection in a principal fibre bundle is proposed. An interrelation between the WE pseudopotentials and fibre bundle connections was pointed out by Hermann (1976). An approach based on fibre bundles was elaborated by Crampin *et al* (1977), Dodd and Gibbon (1978) and Konopelchenko (1979). But these authors proceed from the known linear problem (the generalised Zakharov–Shabat (AKNS) problem (Ablowitz *et al* 1974)), while the WE method is intended primarily for finding such a problem. It should be noted that Morris (1979) found the WE-type structure for some class of non-linear equations with two spatial dimensions.

The present method will be illustrated on an example of a system of equations (Its, cited by Dubrovin *et al* 1976)

$$\begin{aligned}
&\mathrm{i}u_t^{(1)}+u_{xx}^{(1)}-2\mathrm{i}u^{(1)}u_x^{(1)}-2\mathrm{i}u_x^{(2)}=0\\
&\mathrm{i}u_t^{(2)}-u_{xx}^{(2)}-2\mathrm{i}(u^{(1)}u^{(2)})_x=0.
\end{aligned}\tag{1}$$

For this system a linear problem and conservation laws are obtained.

We shall consider a system of non-linear evolution equations in two dimensions (x, t) of the form

$$u_t^{(\sigma)} = K^{(\sigma)}(u, u_x, u_{xx}, \ldots, u_{(n)x}) \qquad \sigma = 1, \ldots, S \tag{2}$$

including derivatives with respect to x up to order n. Here $K^{(\sigma)}$ is some set of (non-linear) functions of indicated variables and S is the number of equations in the system. For simplicity we assume the functions $K^{(\sigma)}$ include the term with the highest derivative additively, i.e. $K^{(\sigma)} = \gamma^{(\sigma)} u_{(n)x}^{(\sigma)} + \mathscr{K}^{(\sigma)}(u, \ldots, u_{(n-1)x})$ where $\gamma^{(\sigma)}$ is a constant. Let us denote $u_x = u_1$, $u_{xx} = u_2, \ldots, u_{(n)x} = u_n$. Then following WE the system (2) can be represented as a set of nS two-forms

$$\begin{aligned} \alpha_1^{(\sigma)} &= \mathrm{d}u^{(\sigma)} \wedge \mathrm{d}t - u_1^{(\sigma)} \, \mathrm{d}x \wedge \mathrm{d}t, \ldots, \alpha_{n-1}^{(\sigma)} = \mathrm{d}u_{n-2}^{(\sigma)} \wedge \mathrm{d}t - u_{n-1}^{(\sigma)} \, \mathrm{d}x \wedge \mathrm{d}t \\ \alpha_n^{(\sigma)} &= \mathrm{d}u_{n-1}^{(\sigma)} \wedge \mathrm{d}t + \gamma^{(\sigma)-1} \, \mathrm{d}u^{(\sigma)} \wedge \mathrm{d}x + \gamma^{(\sigma)-1} \mathscr{K}^{(\sigma)} \, \mathrm{d}x \wedge \mathrm{d}t \end{aligned} \tag{3}$$

which are annulled by a regular two-dimensional solution manifold $S_2(x, t)$ and constitute a closed ideal of forms.

Let there be connected with the system (2) a principal fibre bundle $P(M, \tilde{G})$ (Chern 1956, Sternberg 1964) where M is a base manifold whose every point is represented by an infinite set $z = \{z^\mu\} = (z^1 = x, z^2 = t, z^3 = u^{(1)}, \ldots, z^{2+S} = u^{(S)}, z^{3+S} = u_1^{(1)}, \ldots)$ and $\tilde{G}$ is a structure Lie group. If one represents locally a point $b \in P$ as (z, s) where s stands for coordinates of a group manifold (fibre), then a connection one-form on P is written as (Chern 1956)

$$\omega(b) = \Theta(s) + (\mathrm{Ad}\, s^{-1}) A_\mu(z) \, \mathrm{d}z^\mu. \tag{4}$$

$\Theta(s)$ is a left-invariant one-form satisfying the Maurer–Cartan equation and the functions $A_\mu(z)$ $(\mu = 1, 2, \ldots)$ are defined on M. The summation convention is used. All the terms in (4) have their values in the Lie algebra g of the structure group $\tilde{G}$; in particular, $\omega = \omega^k L_k$, $A_\mu = A_\mu^k L_k$ $(k = 1, 2, \ldots, \dim g)$, ω^k and A_μ^k are scalar-valued forms and functions respectively. L_k are generators of g and $[L_l, L_m] = c_{lm}^k L_k$.

The connection form of the type (4) with $\mu = 1, \ldots, 4$ is used (Konopleva and Popov 1972, Drechsler and Mayer 1977) for a geometrical description of the non-Abelian gauge fields. Bearing in mind this analogy, we shall call the A_μ quasipotentials.

The curvature two-form $\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$ (Chern 1956) for the connection (4) has the form $\Omega = \frac{1}{2}(\mathrm{Ad}\, s^{-1}) F_{\mu\nu} \, \mathrm{d}z^\mu \wedge \mathrm{d}z^\nu$ where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \qquad \partial_\mu \equiv \partial/\partial z^\mu \tag{5}$$

and the commutator is defined as $[A_\mu, A_\nu] = c_{lm}^k A_\mu^l A_\nu^m L_k$.

We introduce now a vector bundle $Q(P)$ associated with the principal fibre bundle $P(M, \tilde{G})$ (Sternberg 1964). Here Q is an N-dimensional vector space in which a linear representation of $\tilde{G}$ acts. The connection in P induces a connection in $Q(P)$ which allows us to define a parallel transport. Namely, two Q vectors $y(z)$ and $y(z + \mathrm{d}z)$ in the points z and $z + \mathrm{d}z$ will be parallel if

$$y(z) - y(z + \mathrm{d}z) = A_\mu(z) y(z) \, \mathrm{d}z^\mu. \tag{6}$$

For brevity we do not make a distinction between A_μ in (4) and its representation in Q in (6).

Finally, a holonomy algebra h of the connection (4) is generated (Loos 1967) by all linear combinations of $F_{\mu\nu}$, $\nabla_\rho F_{\mu\nu}$, $\nabla_\sigma \nabla_\rho F_{\mu\nu}, \ldots$, where ∇_ρ is a covariant derivative,

$\nabla_\rho F_{\mu\nu} = \partial_\rho F_{\mu\nu} - [A_\rho, F_{\mu\nu}]$. In the case where every element of h is a linear combination of $F_{\mu\nu}$ alone, the holonomy algebra is called perfect.

We shall now show that the results of WE follow from the very special choice of a class of the connection forms (4). In fact, let us take

$$A_1 = F(u, u_1, u_2, \ldots) \qquad A_2 = G(u, u_1, u_2, \ldots) \qquad A_3 = A_4 = \ldots = 0. \tag{7}$$

Then we get

$$\Omega = (\operatorname{Ad} s^{-1}) \sum_{\sigma=1}^{S} (F_{u_r^{(\sigma)}}\, \mathrm{d}u_r^{(\sigma)} \wedge \mathrm{d}x + G_{u_r^{(\sigma)}}\, \mathrm{d}u_r^{(\sigma)} \wedge \mathrm{d}t + [F, G]\, \mathrm{d}x \wedge \mathrm{d}t). \tag{8}$$

Here $[F, G] = c^k_{lm} F^l G^m L_k$. The summation over r is taken from 0 to ∞ ($u_0 \equiv u$). $F_{u_r^{(\sigma)}}$ means a partial derivative $\partial F/\partial u_r^{(\sigma)}$.

In a number of papers (Hermann 1976, Crampin *et al* 1977, Crampin 1978, Dodd and Gibbon 1978, Konopelchenko 1979) it has been observed that a given non-linear equation with soliton properties can be connected with the vanishing curvature of some fibre bundle. As a consequence of this observation, we take the curvature form Ω to be a linear combination of the $\alpha_a^{(\sigma)}$'s (3):

$$\Omega = \sum_{\sigma=1}^{S} \sum_{a=1}^{n} \beta_a^{(\sigma)} \alpha_a^{(\sigma)} \tag{9}$$

where the $\beta_a^{(\sigma)}$ are some g-valued functions. On the solution manifold S_2 the curvature Ω vanishes. Then we obtain from (9) a system of equations for the quasipotentials F and G which we shall call the Wahlquist–Estabrook equations:

$$\begin{gathered} F = F(u) \qquad G_{u_{n-1}^{(\sigma)}} = \gamma^{(\sigma)} F_{u^{(\sigma)}} \qquad G_{u_{n-1+i}^{(\sigma)}} = 0 \qquad i = 1, 2, \ldots \\ [F, G] + \mathrm{D}G - \sum_{\sigma=1}^{S} K^{(\sigma)} F_{u^{(\sigma)}} = 0. \end{gathered} \tag{10}$$

Here D is the total derivative $\mathrm{D} = \Sigma_\sigma\, (u_1^{(\sigma)}\, \partial/\partial u^{(\sigma)} + u_2^{(\sigma)}\, \partial/\partial u_1^{(\sigma)} + \ldots)$. The expansion coefficients $\beta_a^{(\sigma)}$ in (9) are expressed in terms of the quasipotential G: $\beta_a^{(\sigma)} = G_{u_a^{(\sigma)}}$.

It should be stressed that, firstly, as distinct from WE, the quasipotentials F and G do not depend on the auxiliary prolongation variables and, secondly, the commutator $[F, G]$ is defined by the structure constants. Hence, for the existence of the non-Abelian WE structure the structure group $\tilde{G}$ must be non-Abelian.

With a glance at the restriction (7), the parallel transport equations which follow from (6) take the form

$$y_x = -Fy \qquad y_t = -Gy. \tag{11}$$

The interpretation of the WE pseudopotentials as coordinates of a representation space of some group was proposed by Corones *et al* (1977). Equation (11) provides the explicit proof of this fact.

Further analysis of the WE equations (10) demands a knowledge of concrete expressions for the functions $K^{(\sigma)}$.

Let us return to the system (1). This system belongs to the class of equations of the type (2) with $S = 2$, $n = 2$, $K^{(1)} = \mathrm{i}u_2^{(1)} + 2u^{(1)}u_1^{(1)} + 2u_1^{(2)}$, $K^{(2)} - \mathrm{i}u_2^{(2)} + 2(u^{(1)}u_1^{(2)} + u_1^{(1)}u^{(2)})$. The WE equations (10) have the form

$$F = F(u) \qquad G = G(u, u_1) \qquad G_{u_1^{(1)}} = \mathrm{i}F_{u^{(1)}} \qquad G_{u_1^{(2)}} = -\mathrm{i}F_{u^{(2)}}$$

$$[F, G] + u_1^{(1)} G_{u^{(1)}} + u_1^{(2)} G_{u^{(2)}} + 2\mathrm{i}G_{u_1^{(1)}}(u^{(1)}u_1^{(1)} + u_1^{(2)}) - 2\mathrm{i}G_{u_1^{(2)}}(u^{(1)}u_1^{(2)} + u_1^{(1)}u^{(2)}) = 0.$$

Following the well known procedure for solving similar equations and introducing the individual notations $u^{(1)}=u$, $u^{(2)}=v$ we obtain

$$
\begin{aligned}
F &= -\mathrm{i}(uvX_1+uX_2+vX_3+X_4)\\
G &= -\mathrm{i}\{[v(2u^2+v)+\mathrm{i}(u_1v-uv_1)]X_1+(2v+u^2+\mathrm{i}u_1)X_2\\
&\qquad +(2uv-\mathrm{i}v_1)X_3+X_5+uvX_6+uX_7-vX_8\}.
\end{aligned}
\tag{12}
$$

Here $X_\alpha = X_\alpha^k L_k$ are g-valued constants of integration and X_α^k are numbers. These X_α's define an algebraic structure

$$
\begin{aligned}
&[X_1, X_\alpha]=0 \qquad (\alpha\neq 5) \qquad [X_3,X_6]=[X_4,X_5]=0\\
&[X_2,X_3]=X_6 \qquad [X_2,X_4]=X_7 \qquad [X_3,X_4]=X_8\\
&[X_2,X_7]=X_7 \qquad [X_3,X_8]=-2X_6\\
&[X_2,X_5]+[X_4,X_7]=0 \qquad [X_3,X_5]-[X_4,X_8]=2X_7\\
&[X_1,X_7]+[X_2,X_6]+X_6=0 \qquad [X_1,X_5]-[X_2,X_8]+[X_3,X_7]+[X_4,X_6]=2X_6.
\end{aligned}
\tag{13}
$$

Here the commutators are again defined via the structure constants: $[X_\alpha, X_\beta]= c_{lm}^k X_\alpha^l X_\beta^m L_k$.

The structure (13) does not close itself into a finite-dimensional Lie algebra. To close this structure we consider now the holonomy algebra of the connection (4) with the restriction (7). The key step for closing uniquely the structure (13) is to demand that the holonomy algebra be non-Abelian and *perfect*, i.e. it must be generated entirely by $F_{\mu\nu}$. In the case under consideration the quantity $F_{\mu\nu}$ (5) has the following non-zero components:

$$
\begin{aligned}
&F_{12}=[F,G]=-u_1G_u-v_1G_v-2\mathrm{i}G_{u_1}(uu_1+v_1)+2\mathrm{i}G_{v_1}(uv_1+u_1v)\\
&F_{13}=-\partial_uF \qquad F_{14}=-\partial_vF \qquad F_{23}=-\partial_uG \qquad F_{24}=-\partial_vG\\
&F_{25}=-\partial_{u_1}G \qquad F_{26}=-\partial_{v_1}G.
\end{aligned}
$$

With provision for explicit expressions for F and G (12) and for commutators with X_1 the holonomy algebra is generated by X_2, X_3, X_6, X_7, X_8. In other words, we demand that these elements generate a non-Abelian Lie algebra, i.e. $[X_2, X_8]= a_2X_2+a_3X_3+a_6X_6+a_7X_7+a_8X_8$, etc. After tedious but straightforward calculation using the Jacobi identities we obtain the following Lie algebra ($\lambda\equiv -a_6$, X_1 commutes with all X_α's)

	X_3	X_4	X_5	X_6	X_7	X_8
X_2	X_6	X_7	λX_7	$-X_6$	X_7	$-\lambda X_6$
X_3		X_8	λX_8	0	$-\lambda X_6+X_8$	$-2X_6$
X_4			0	X_8	$-\lambda X_7$	$-2X_7+\lambda X_8$
X_5				λX_8	$-\lambda^2 X_7$	$-2\lambda X_7+\lambda^2X_8$
X_6					$\lambda X_6 - X_8$	$2X_6$
X_7						$-\lambda^2X_6-2X_7+\lambda X_8$.

This Lie algebra represents the unique possibility of closing the structure (13) compatible with the perfectness of the non-Abelian holonomy algebra.

It should be stressed that the proposed way of closing is a purely computational one with a well defined line of attack. If it leads to closing the structure, then the corresponding finite-dimensional algebra is unique. If such a holonomy algebra does not exist, other methods ought to be used.

It is easy to see that the algebra obtained is $\mathrm{sl}(2)+Z^{(4)}$ where $Z^{(4)}$ is the four-dimensional centre, and the X_α are expressed in terms of a basis Y_3, $Y_\pm$ of sl(2) with commutators $[Y_3, Y_\pm]=\pm 2Y_\pm$, $[Y_+, Y_-]=Y_3$ as follows:

$$X_2=-\tfrac{1}{2}Y_3 \qquad X_3=-Y_+ \qquad X_4=\tfrac{1}{2}\lambda Y_3+Y_- \qquad X_5=\tfrac{1}{2}\lambda^2 Y_3+\lambda Y_-$$

$$X_6=Y_+ \qquad X_7=Y_- \qquad X_8=\lambda Y_+-Y_3.$$

Realising sl(2) by means of 2×2 matrices, we find a 2×2 matrix realisation of the quasipotentials F and G:

$$F=-\mathrm{i}\begin{pmatrix}-\frac{1}{2}(u-\lambda) & -v\\ 1 & \frac{1}{2}(u-\lambda)\end{pmatrix} \qquad G=-\mathrm{i}\begin{pmatrix}-\frac{1}{2}(u^2-\lambda^2+\mathrm{i}u_1) & -v(u+\lambda)+\mathrm{i}v_1\\ u+\lambda & \frac{1}{2}(u^2-\lambda^2+\mathrm{i}u_1)\end{pmatrix}. \tag{14}$$

By virtue of (11) we therefore obtain the associated linear problem. Evidently, the dimension of the linear problem depends on the dimension of the sl(2) representation.

The linear spectral problem $y_x=-Fy$ with F given by (14) does not belong to the AKNS type. Therefore, to obtain conservation laws there must be some modification to the results of Wadati *et al* (1975) concerning the derivation of conservation laws from the known linear problem. First of all, we write equation (11) in the Riccati form ($\Gamma=y_1/y_2$):

$$\mathrm{i}\Gamma_x=v+(u-\lambda)\Gamma+\Gamma^2 \qquad \mathrm{i}\Gamma_t=v(u+\lambda)-\mathrm{i}v_x+(u^2-\lambda^2+\mathrm{i}u_x)\Gamma+(u+\lambda)\Gamma^2.$$

It can be shown that the following relation holds:

$$\frac{\partial}{\partial t}\Gamma=\frac{\partial}{\partial x}(-v+(u+\lambda)\Gamma). \tag{15}$$

Expanding Γ in a series $\Gamma=\Sigma_{n=1}^{\infty} f_n\lambda^{-n}$ we find from $\Gamma=(v+u\Gamma+\Gamma^2-\mathrm{i}\Gamma_x)\lambda^{-1}$ a recurrence relation for f_n:

$$f_{n+1}=v\delta_{0n}+uf_n+\sum_{k=1}^{n-1} f_kf_{n-k}-\mathrm{i}f_{nx}.$$

The first three densities are the following:

$$f_1=v \qquad f_2=uv-\mathrm{i}v_x \qquad f_3=u^2v-2\mathrm{i}uv_x-\mathrm{i}u_xv+v^2-v_{xx}.$$

Then the conservation laws follow from (15):

$$\frac{\partial}{\partial t}f_n=\frac{\partial}{\partial x}(uf_n+f_{n+1}) \qquad n=1,2,\ldots.$$

Thus far, the WE method has been applied to equations whose solutions go to zero sufficiently fast at infinity. Here we point out, on an example of the Korteweg–de Vries (KdV) equation, an interrelation between the quasipotentials F and G and the problem of finding solutions periodic in x.

The quasipotentials for the KdV equation $u_t+u_{xxx}+12uu_x=0$, as can be shown by the present method, have the form

$$F=2^{1/2}\begin{pmatrix}0 & -1\\ \frac{1}{2}\lambda+u & 0\end{pmatrix} \qquad G=2^{1/2}\begin{pmatrix}-2^{1/2}u_1 & 4(u-\lambda)\\ -u_2-4u^2+2\lambda^2+2u\lambda & 2^{1/2}u_1\end{pmatrix}$$

and the linear spectral problem is the Schrödinger equation $y_{xx}+(\lambda+2u)y=0$. Let the potential u be periodic in x with period T, i.e. there exists a monodromy operator $(\hat{T}y)(x)=y(x+T)$. Let us fix a point x_0 and consider a basis in a space of solutions of the Schrödinger equation with the properties (Dubrovin *et al* 1979)

$$c=1 \qquad c_x=0 \qquad s=0 \qquad s_x=1 \qquad \text{at } x=x_0.$$

The monodromy operator in this basis is a 2×2 matrix $\hat{T}c=\alpha_{11}c+\alpha_{12}s$, $\hat{T}s=\alpha_{21}c+\alpha_{22}s$. Then a dependence of $\hat{T}$ on the parameter x_0 is given by the quasipotential F and time dependence is governed by the quasipotential G:

$$\partial\hat{T}/\partial x_0=[F,\hat{T}] \qquad \partial\hat{T}/\partial t=[G,\hat{T}].$$

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